Bootstrap proof of first-order transitions?

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1 Introduction

Some phase transitions are continuous, others first-order. Landau's criterion for a continuous transition says that there should be no invariant cubic term in the effective Hamiltonian. In group theory language this reads

$$[R^3] \not\ni E, \tag{1}$$

where R is the representation in which the order parameter transforms, $[R^3]$ is the symmetric part of R^3, and E is the unit representation.

Landau's criterion correctly predicts transition order in many cases, but not always. E.g., there are continuous transitions which violate it, such as the 3-state Potts model in 2d. From RG perspective, this may be explained by saying that Landau's criterion is not applicable if the cubic operator transforming as $[R^3]$ is irrelevant.

There also exist transitions which satisfy Landau's criterion and yet they are first-order. From RG perspective, this may be explained if there is no stable fixed point (because the quartic couplings flow to infinity or to the region where the potential is unbounded from below). Such a first-order transition is called "fluctuation-driven", to distinguish from transitions which are trivially first-order because they violate Landau.

From the **bootstrap perspective**, transition must be first-order if there is no CFT with the required symmetry properties, and it may be continuous if the CFT exists (and if the microscopic model lies in the basin of attraction).

This is how bootstrap contributed to clarifying the nature of the deconfined criticality transition (see [1], section V.E.4, for a review). While a continuous transition was conjectured based on Monte Carlo simulations, bootstrap showed that there is no unitary CFT with the needed symmetry and scaling dimensions, and so the transition must be weakly 1st order.[1]

Here we will discuss two other examples where transition is known to be 1st order, and it would be interesting to get a bootstrap proof of this fact.

2 Q-state Potts model in 3d

Lattice simulations show that Q-state nearest-neighbor ferromagnetic Potts model with $Q \geqslant 3$ has a 1st-order transition in 3d. (In agreement with Landau's criterion.) Can we explain this via bootstrap?

This is of interest for the condensed matter and mathematical physics communities (Max Metlitski, Hugo Duminil-Copin, personal communications). Perthaps a fixed point exists but the nearest-neighbor Potts model lies outside of its basin of attraction? A bootstrap proof would exclude this.

From the bootstrap perspective, one would have to prove that there is no unitary 3d CFT with S_Q global symmetry and some assumptions on the spectrum of local operators (e.g. there must be only one singlet scalar). This problem is open. A few people told me they tried, although I don't remember all of their names.[2] Please let me know if you worked/are working on this.

1. Update (July 2025). This question has been recently reopened. A tricritical point is another seriously considered possibility. It is compatible with the conformal bootstrap constraints [2].

2. Shai Chester, Alessandro Vichi, Bernardo Zan…

One possible strategy to attack this question for $Q=3$ (Shai Chester, personal communication) is to start in 2d and find a set of constraints which puts the 2d 3-state Potts CFT in a bootstrap island. Unfortunately this set of bootstrap constraints is not easy to find. If this could be done, one could increase d gradually and see if at some d_c the island disappears.[3]

Another possibility is to start directly in 3d and consider Q-state Potts model with Q large. First-order transition is known to be stronger for large Q. Perhaps the bootstrap problem will also be easier for large Q?

It's also interesting to consider a more general problem relaxing symmetry for $Q=3$ from S_3 to $\mathbb{Z}_3$ (Max Metlitski, personal communication).

3 Antiferromagnets

Phase transitions in antiferromagnets may have order parameters with $n \geqslant 4$ components, and there are many situations where RG predicts a fluctuation-driven 1st-order phase transition, in agreement with experiment. An exhaustive study was done in 1976 Mukamel and Krinsky [6],[7]. I am not aware of any bootstrap work.

Consider e.g. the case discussed in [6], p.5071 section "Type-I antiferromagnets, $\vec{m} \perp \vec{k}$". The Landau-Ginzburg theory has a 6-component order parameter $\varphi = (\phi_1, \phi_2, \phi_3, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$. Denoting $x_i = \phi_i^2$, $y_i = \bar{\phi}_i^2$, the quartic potential has the form

$$\begin{aligned} V = \; & r(x_1 + x_2 + x_3 + y_1 + y_2 + y_3) \\ & + u_1(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2) \\ & + u_2(x_1 y_1 + x_2 y_2 + x_3 y_3) \\ & + u_3(x_1 x_2 + x_1 x_3 + x_2 x_3 + y_1 y_2 + y_1 y_3 + y_2 y_3) \\ & + u_4(y_1 x_2 + y_2 x_3 + y_3 x_1) \\ & + u_5(x_1 y_2 + x_2 y_3 + x_3 y_1) \end{aligned}$$

In addition to the $(\mathbb{Z}_2)^6$ transformations which flip the individual signs of components of φ, the potential is left invariant by two permutations (given in the cycle representation)

$$p_1 = (x_1\, y_2)(x_2\, y_1)$$
$$p_2 = (x_1\, x_2\, x_3)(y_1\, y_2\, y_3)$$

which generate a subgroup $G \subset S_6$ (it should be easy to figure out which one using GAP:). The full symmetry group is then $G \ltimes (\mathbb{Z}_2)^6$. One-loop beta-functions for this theory are given on p. 5081 of [7], and while there are many fixed points, all of them are shown to be unstable. This suggests that the phase transition must be first-order. This agrees with experiments in the uranium dioxide UO_2, and with a Monte Carlo study in a related model in the same universality class [8].

It would be great to find bootstrap evidence for this fact, by excluding unitary CFTs with only one relevant singlet and $G \ltimes (\mathbb{Z}_2)^6$ symmetry. This appears hard in full generality, but perhaps it can be done for Δ_φ not much above 0.5? Or for the central charge not much different from that of 6 free scalars?

Same question can be asked for many other transitions discussed in [6],[7].

Bibliography

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3. Update (July 2025). This has been attempted recently in [3]. However isolation into an island was not obtained, and the approach to d_c did not follow the expected square-root behavior, so more work is needed. See [4] for a discussion. See also [5] for related work of studying disappearance of bootstrap islands as a function of N, in a model with $O(N) \times O(2)$ global invariance.

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