Bounding Operator Dimensions from Conformal Bootstrap

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SCUOLA NORMALE SUPERIORE PISA

Based on: 0807.0004

with

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Pheno motivation [Luty, Okui 2004]

A Higgs sector with conformal symmetry
at E>>TeV and

(1) $[H] < 1+1/\text{few}$

(2) $[H^\dagger H] > 4 - 1/\text{few}$ large anomalous dim

(2) => moderate hierarchy (e.g. up to ~1000 TeV) can be generated without finetuning

(1) => strong-coupling top-quark Yukawa scale above ~1000 TeV
=> robust suppression of FCNC, CP violation

Do such 4D CFTs exist?

1. Scalar O(N) models $\lambda(\phi^2)^2$ have fixed points only in (4-ϵ) (Wilson-Fischer)

2. Belavin-Migdal-Banks-Zaks fixed points (N_F/N~11/2, N>>1) coupled to $\boldsymbol{\phi}$ via Yukawa interactions => CFTs with

   $[\boldsymbol{\phi}]$=1+O(1/N), $[\boldsymbol{\phi}^{2}]$=2+O(1/N)

3. General feature of large N CFT (in particular with gravity duals):

   [**O**^{2}]=2[**O**]+O(1/N)

2D and 3D examples

show that $\gamma_{\phi^2} >> \gamma_\phi$ is not impossible.

Ising model: $\boldsymbol{\sigma \times \sigma = 1 + \varepsilon}$

2-dimensions (Onsager)	$[\sigma] = \mathbf{1/8}, \quad [\varepsilon] = \mathbf{1}$
3-dimensions (ϵ- and high-T expansions, Monte-Carlo)	$\gamma_\sigma \approx \mathbf{0.02}, \quad \gamma_\varepsilon \approx \mathbf{0.4}$

Formulation of the problem

Take any unitary 4D CFT

Take any scalar primary ϕ (hermitean)

Call ϕ^2 lowest dimension scalar in OPE:

$$\phi \times \phi = \mathbf{1} + \phi^2 + \ldots$$

Suppose, for a fixed $d = [\phi]$ *,*
I want to have $\Delta = [\phi^2]$ *as high as possible.*
Is there a limit on how high I can get?

Remarks

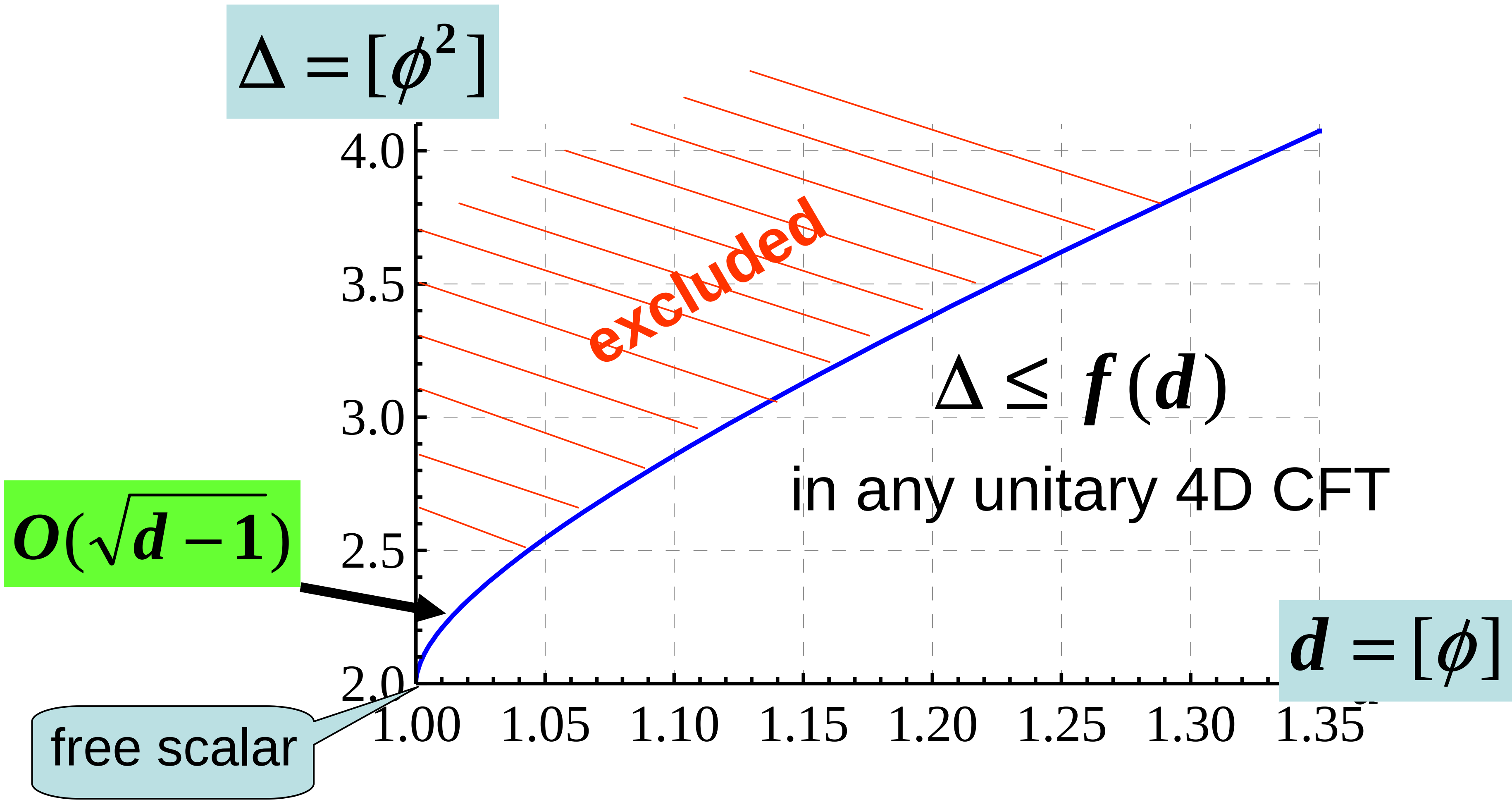
1. It's a simplified version of the pheno problem:
   no classification under global symmetry
   (For conformal Higgs, $[H^\dagger H]$ is the lowest dimension **singlet)**

2. Question makes sense for any $d=[\phi]$
   we are particularly interested in $d<1+1/\text{few}$

3. Can be asked in any spacetime dimensions

Answer preview

Model-independent *upper* bound:



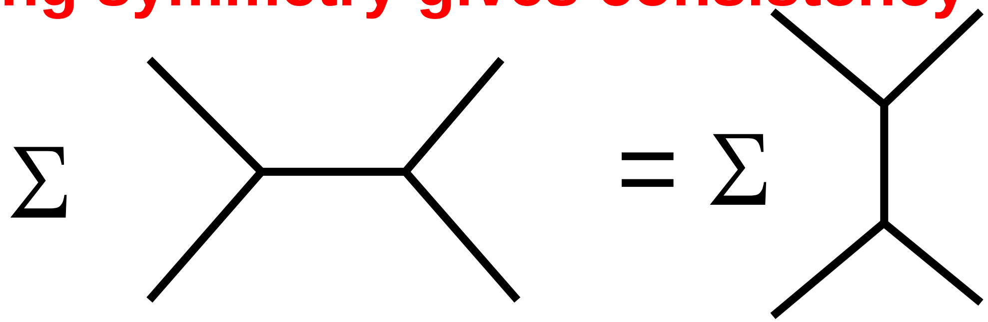
Method: Conformal bootstrap

CFT is characterized by

- spectrum of operator dimensions and spins $\{\Delta, \ell\}$
- structure constants (3-point functions) $\langle O_a O_b O_c \rangle \sim \lambda_{abc}$

Any n-point function can be computed by OPE

Crossing symmetry gives consistency condition:



- infinite system of linear equations for **squares** of structure functions
- if imposed for all fields, is supposed to severely constraint (fix?) the theory.

An old idea...

Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences

(Submitted July 9, 1973)

Zh. Eksp. Teor. Fiz. 66, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the S-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

RODUCTION

ecent years, the hypothesis of the conformal in-
:e of strong interactions at distances much
· than 10^{-14} cm has been put forward and analyzed
il (see the reviews[1,2]). It has been shown that

may be hoped that, as in a theor
combination of the completeness
causality condition will give a d
tions, sufficient for the determi
functions.

CFT Ingredients

- OPE with full spectrum and at finite separation
- conformal block decomposition of 4-pt function
- unitarity bounds for higher-spin primaries
- reality conditions for the OPE coefficients

OPE

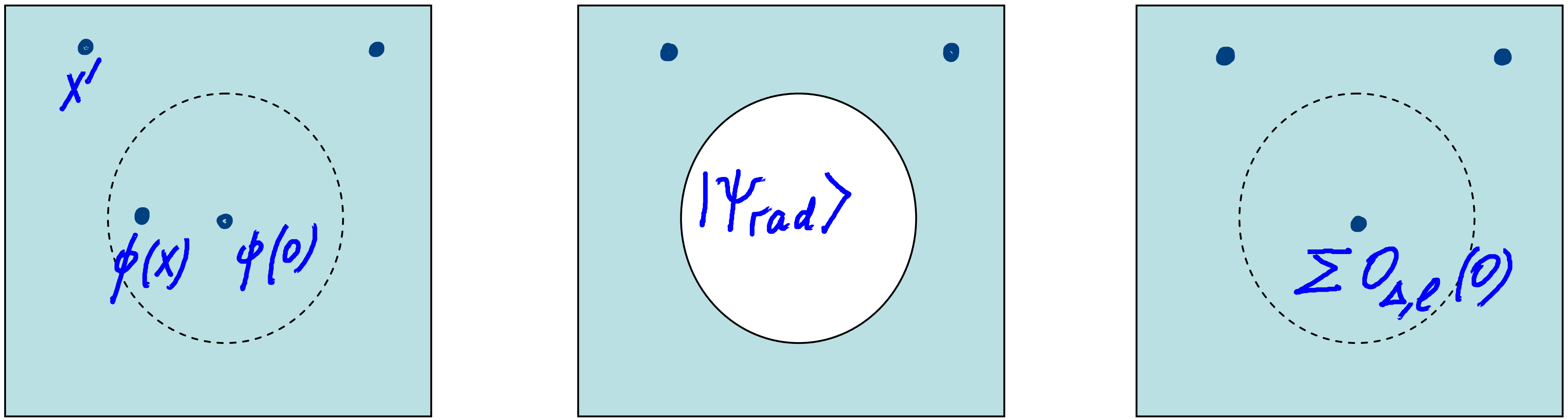
$$\phi(x)\phi(0) = \frac{1}{x^{2d}} + \sum_{l=0,2,4\ldots} \sum_{\Delta} \lambda_{\Delta,l} [C(x)O(0) + \ldots]$$

descendants

NB. Only even spins appear

Convergence at finite separation:

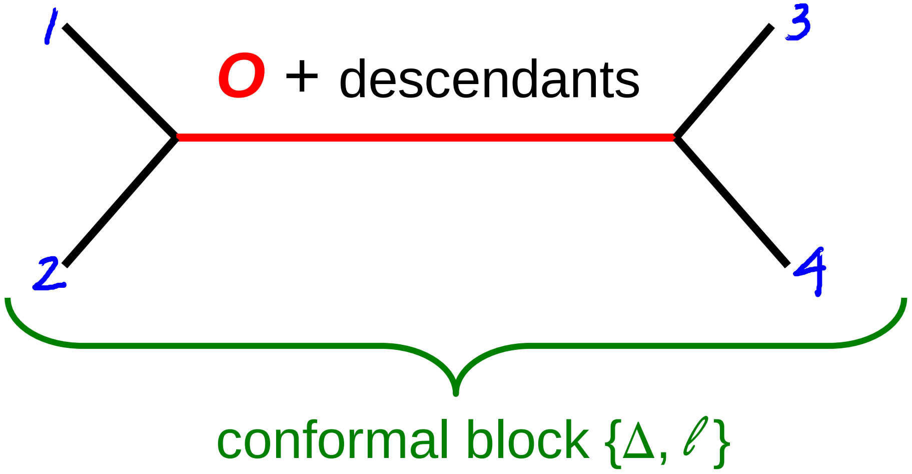
should converge if no other operators with |x'|<|x|



Conformal block decomposition

$$\langle \phi(x_1)\,\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2d}|x_{34}|^{2d}}$$

$$\langle \phi\phi\phi\phi \rangle = \sum_{O}$$



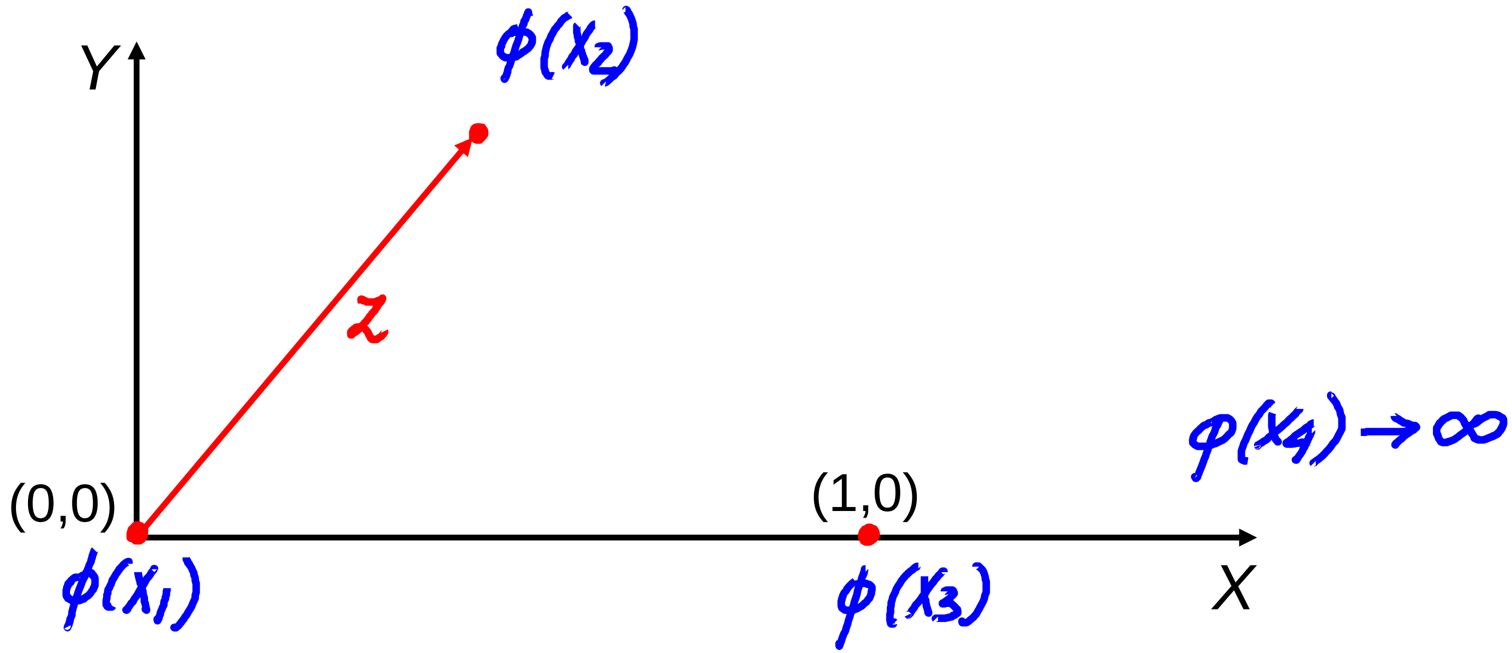
$$g(u,v) = 1 + \sum_{\Delta,l} (\lambda_{\Delta,l})^2 g_{\Delta,l}(u,v)$$

4D Conformal Blocks [Dolan, Osborn, 2001]

$$g_{\Delta,l}(u,v) = \frac{z\bar{z}}{z-\bar{z}}[\,f_{\Delta+l}(z)\,f_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})]$$

$$f_{\beta}(z) = z^{\beta/2}\,{}_2F_1(\frac{\beta}{2},\frac{\beta}{2},\beta;z)$$

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$$



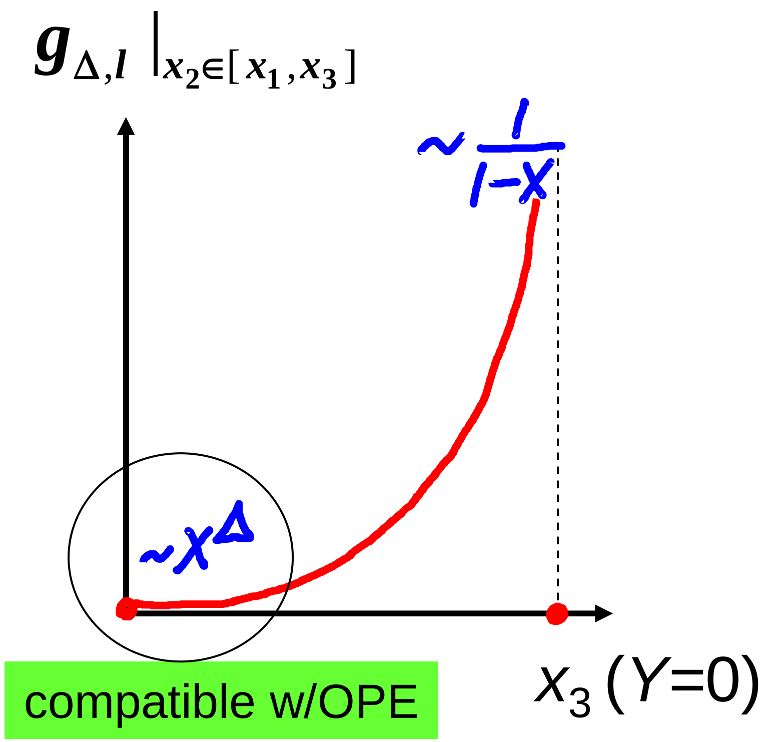
Convergence of Conf. block decomposition

$$g(u,v) = 1 + \sum_{\Delta,l} \left(\lambda_{\Delta,l}\right)^2 g_{\Delta,l}(u,v)$$

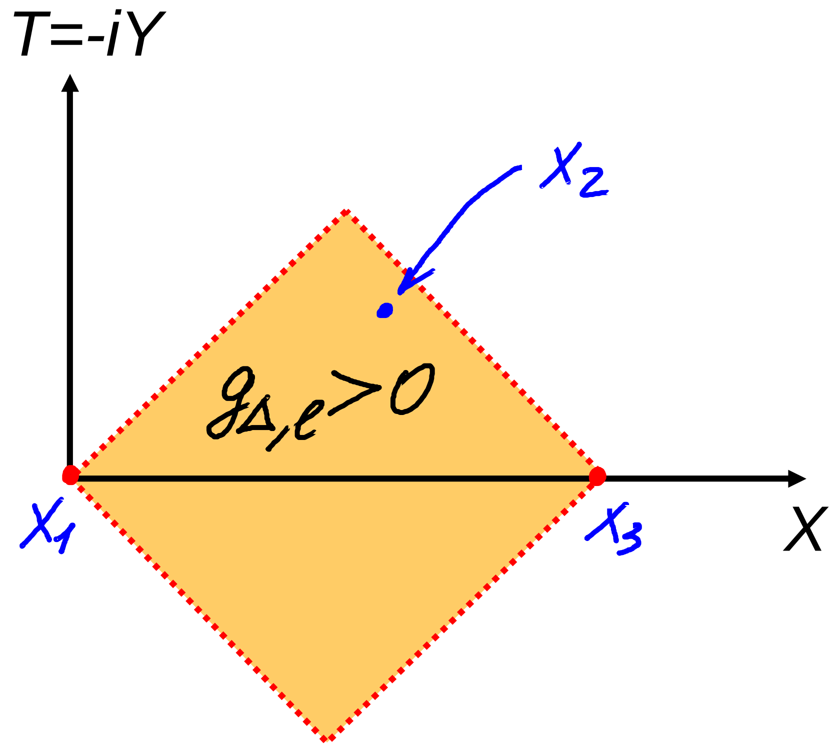
positive coefficients

positive functions

Between x1 and x3:



In spacelike diamond:



Enforcing crossing symmetry

$$g(u,v) = 1 + \sum_{\Delta,l} \left(\lambda_{\Delta,l}\right)^2 g_{\Delta,l}(u,v)$$

- compatible with OPE in 12 and 34

$$\langle \phi\phi\phi\phi \rangle = \frac{g(u,v)}{|x_{12}|^{2d}|x_{34}|^{2d}} = \frac{g(v,u)}{|x_{13}|^{2d}|x_{24}|^{2d}}$$

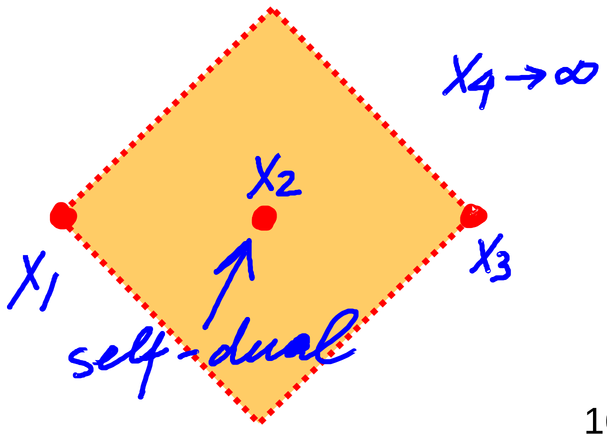
1-3 crossing

$$g(u,v) = \frac{u^d}{v^d} g(v,u)$$

converges fastest for x2->x1

converges fastest for x2->x3

Best to work near symmetric point: **x2=(x1+x3)/2**



Sum rule

Separate unit operator from the rest:

$$\sum_{\Delta,l} \lambda_{\Delta,l}^2 F_{d,\Delta,l}(u,v) \equiv 1$$

$$F_{d,\Delta,l}(u,v) := \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}$$

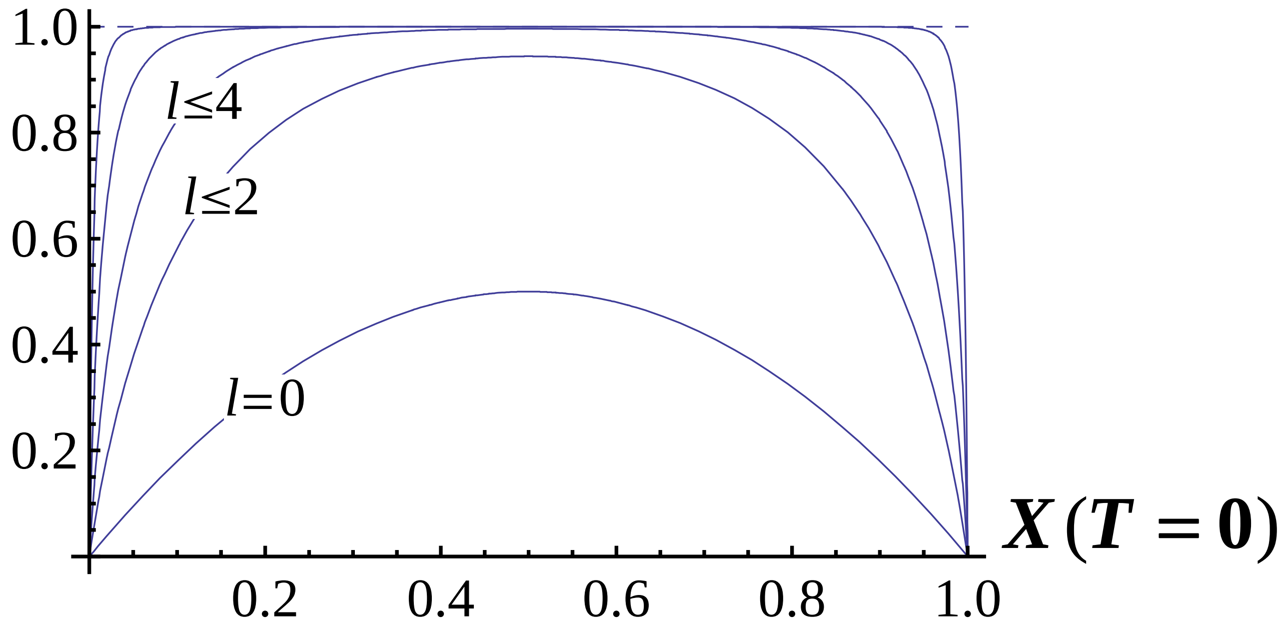
Taylor-expand SUM RULE in **z, zbar** near symmetric point => infinite system of linear equations for $\lambda_{\Delta,l}^2$

Sum rule convergence in free scalar theory

$$\phi \times \phi = \sum_{l=2n} \phi \overleftrightarrow{\partial}^{2n} \phi$$

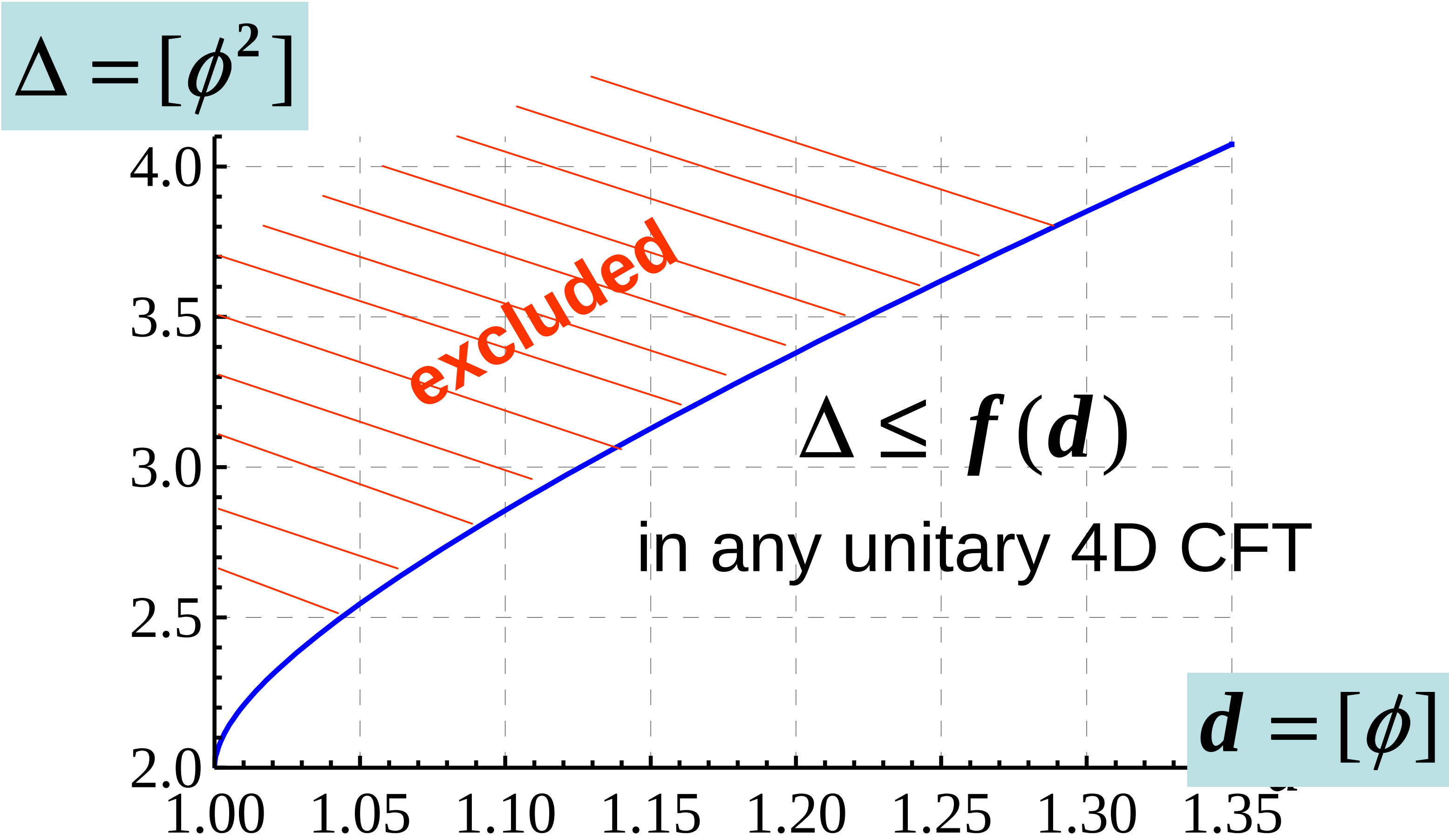
twist 2 fields only

$$\lambda_l^2 = 2^{l+1} \frac{(l!)^2}{(2l!)^2}$$



Monotonic convergence

Back to the result:



Idea

1) Fix ***d***

2) Show that the sum rule:

$$\sum_{\Delta,l} \lambda^2_{\Delta,l} F_{d,\Delta,l}(u,v) \equiv 1$$

cannot be satisfied with **any** $\lambda^2_{\Delta,l} \geq 0$

if the summation is limited to

$$\Delta \geq \Delta_{\min} \quad (l = 0)$$
$$\Delta \geq l + 2 \quad (l = 2,4\ldots)$$

3) The smallest $\Delta_{\min}$ that you can find gives a model-independent upper bound on $[\phi^2]$

A toy example

$$\sum_{\Delta,l} \lambda_{\Delta,l}^2 F_{d,\Delta,l}(u,v) \equiv 1$$

$$\Delta \geq \Delta_{\min} \quad (l=0)$$
$$\Delta \geq l+2 \quad (l=2,4\ldots)$$

Imagine that *2nd derivative* (in a fixed direction) of all F's, for shown restrictions on Δ, is *positive* at a given point

Then we are done! No matter what $\lambda_{\Delta,l}^2 \geq 0$ are, you cannot find a solution

This is exactly what happens. E.g. for **d~1.1** we have

$$F''_{XX} > 0,$$
$$\Delta \geq l + 2 \quad (l = 2,4\ldots)$$
$$\Delta \geq 3.9 \quad (l = 0)$$

@ symmetric point

=> in any CFT with $[\phi]=1.1$, necessarily $[\phi^2]<3.9$

NB: Bound obtained just by using 2nd X-derivative is not very good (it does not even go to 2 as d->1)

General method

$$\sum_{\Delta,l} \lambda^2_{\Delta,l} F_{d,\Delta,l}(u,v) \equiv 1$$

$$\Delta \geq \Delta_{\min} \quad (l=0)$$
$$\Delta \geq l+2 \quad (l=2,4\ldots)$$

1. Look for a differential operator:

$$D[f] = \sum c_{m,n} \partial_X^m \partial_T^n f \quad \text{(no constant term)}$$

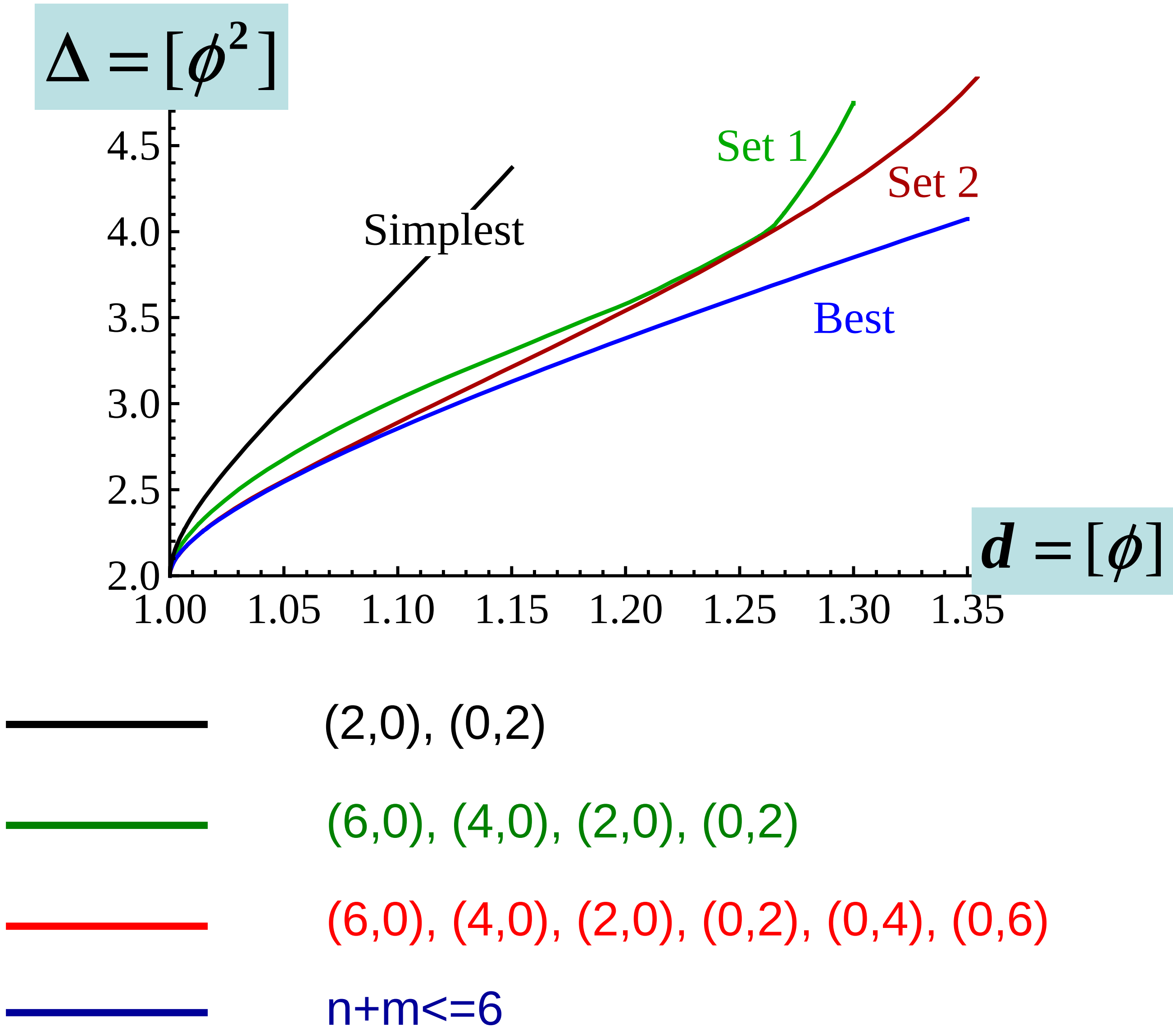
such that $D[F] > 0$ @ symmetric point

Linear Programming problem for $c_{m,n}$

Since $D[1] = 0$ the sum rule cannot be solved for any $\lambda^2_{\Delta,l} \geq 0$

2. Find the smallest $\Delta_{\min}$ for which D[f] exists

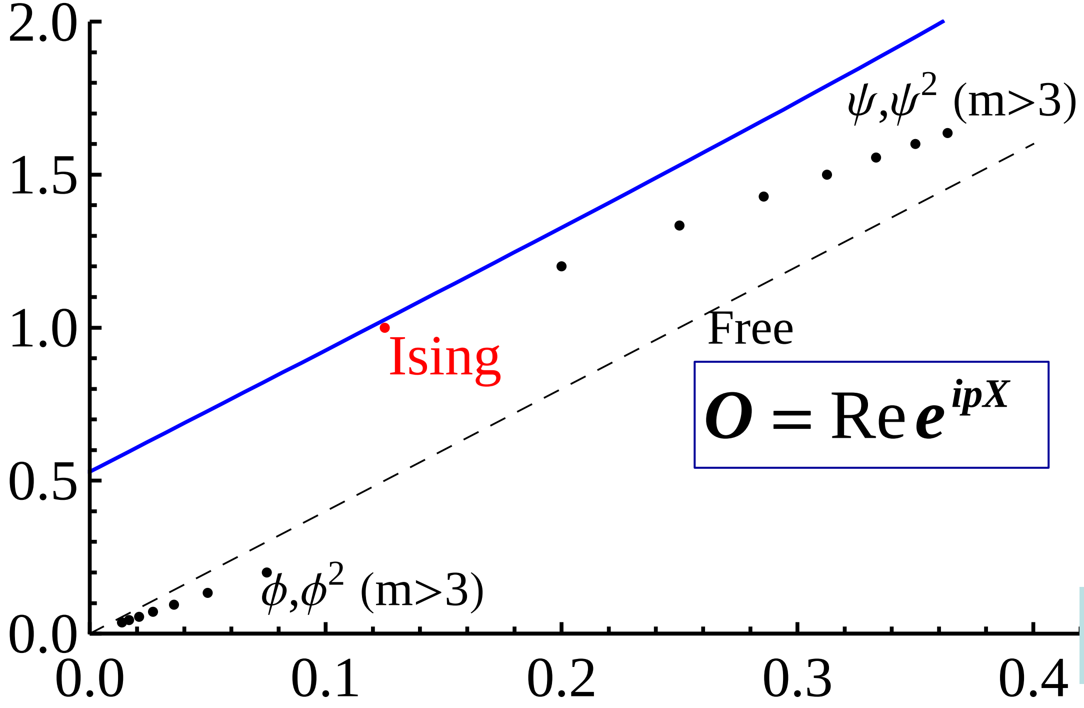
Method gives a sequence of improving bounds
as more and more derivatives are allowed to appear in D[f]



Analogous bound in 2d and checks

$\psi = O_{1,2} \quad \psi^2 = O_{1,3}$

$\Delta = [\phi^2]$



$O = \mathrm{Re}\, e^{ipX}$

$d = [\phi]$

$\phi = O_{2,2} \quad \phi^2 = O_{3,3}$

Conclusions & Future directions

Finally, we are starting to see the power of conformal bootstrap in constraining unitary CFTs.

Many possible extensions and applications.

Also to perturbative string theory,
via the 2d variant of the bound (explored very little)

Extending analysis to 3d?

difficulty: finding 3d conformal blocks

(in odd dim's conformal blocks do not factorize as f(z)f(zbar))

Non-trivial extension for globally-symmetric case?

$$\phi_a \times \phi_b = \delta_{ab}(1 + O^{(1)}) + O^{(2)}{}_{ab} + \dots$$

$$\dots \supset J^{\mu}{}_{ab}$$

-two inequivalent crossing-symmetric 4-pt functions:

$$\langle \phi_1\phi_1\phi_1\phi_1 \rangle \quad \langle \phi_1\phi_2\phi_1\phi_2 \rangle$$

-OPE contains singlets and symmetric-traceless tensors (*even spin*); antisymmetric tensors (*odd spin*)

Can one bound $[O^{(1)}]$ in a model-independent way?

-Paradox in 4-epsilon dimensions

Naive extrapolation of our 4d bound:

$$\gamma_{\phi^2} \leq 1.79\sqrt{\gamma_\phi} \qquad (\gamma_\phi << 1)$$

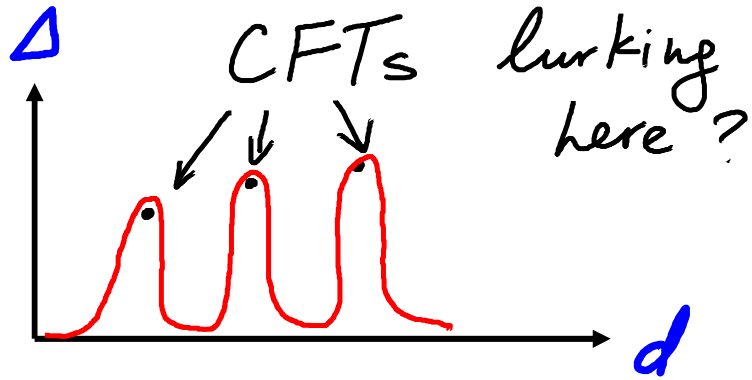
to 4-epsilon is in contradiction with Wilson-Fischer fixed point anomalous dimensions for N=1,2:

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\varepsilon^2$$

$$\gamma_{\phi^2} \equiv \gamma_T = \frac{2}{N+8}\varepsilon$$

- **What is the limiting shape of the bound?**

monotinically converges (to zero?)



limiting delta-function shape

Backup slides

Basic ingredients

Full OPE

$$\phi \times \phi = 1 + \sum_{l=0,2,4\dots} \sum_{\Delta} \lambda_{\Delta,l} O_{\Delta,l}$$

hermitean primaries

even spins

Unitarity bounds

$$\Delta \geq 1 \quad (l = 0)$$

$$\Delta \geq l + 2 \quad (l = 2,4\dots)$$

Reality conditions

real structure constants

$$Z_\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$

$$\left\langle \phi(x_1)\, \phi(x_2)\, O_{\mu_1\dots\mu_l}(x_3) \right\rangle = \frac{\lambda_{\Delta,l}}{|x_{12}|^{2d-\Delta+l} |x_{13}|^{\Delta-l} |x_{23}|^{\Delta-l}} Z_{\mu_1} \dots Z_{\mu_l}$$