Tensor network RG search for critical fixed points: Status report

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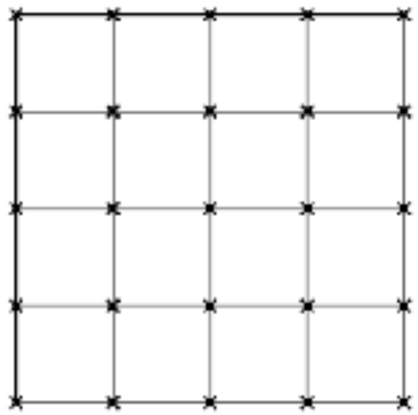
Joint work with **Tom Kennedy** and **Nikolay Ebel**


QFT at the crossroads, Roma La Sapienza, 15-17 September 2025

Basic problems



Stat. phys. Lattice model in D dims
(per se or as regularization of QFT)

- D.o.f. could be discrete (Ising,Potts) or continuous (XY)

- Continuous or gauge symmetries

Typical questions:

Qualitative

-What are the phases?
-What is the nature of phase transition separating them?

Quantitative

-Can one compute, with some accuracy, free energy?
-Can one compute, with some accuracy, critical exponents?

Suff. different…

Renormalization group

- Series of steps increasing lattice spacing $a \to ba \to \ldots \to b^n a \to \ldots$

  preserving partition function (up to a const) $Z_n = e^{\text{Vol} \times g_n} Z_{n+1}$

- (Discrete) flow in the space of Hamiltonians $H_0 \to H_1 \to \ldots$

Answers to typical questions:

- **Phase diagram:** study fixed points of RG map & basin of attraction

- **Free energy** (per site) computed iterating $f(H_n) = g_n + \frac{1}{b^D} f(H_{n+1})$

$$=> \qquad f = \sum_{n=0}^{\infty} \frac{g_n}{b^{Dn}}$$

- **Critical exponents** <=> eigenvalues of the RG map Jacobian at a fixed point

$$\nabla R(H_*)$$

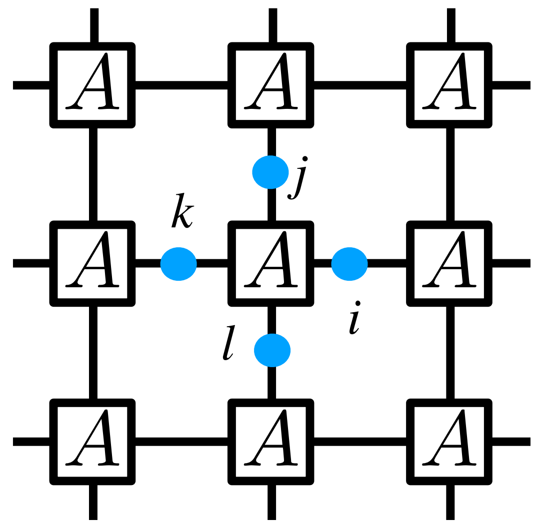
These ideas are >50 years old and no doubt correct. But good/practical implementation is lacking.

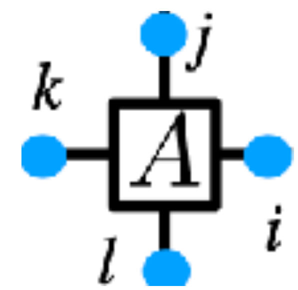
Tensor RG

Levin-Nave PRL 2006

1. Rewrite Z as a tensor network contraction

$$\sum_{\{s\}} e^{H[\{s\}]} =$$





indices = degrees of freedom

i,j,k,l =1… χ

<u>Exercise:</u> Do this for NN Ising model, with $\chi = 2$.

$\chi = \infty$ is OK as long as $\|A\| = \left(\sum_{i,j,k,l} |A_{ijkl}|^2\right)^{1/2} < \infty$

E.g. XY model needs $\chi = \infty$

2. RG map: $\mathscr{R} : A \mapsto A'$ so that $Z(A, N \times M) = Z(A', \frac{N}{b}, \frac{M}{b})$

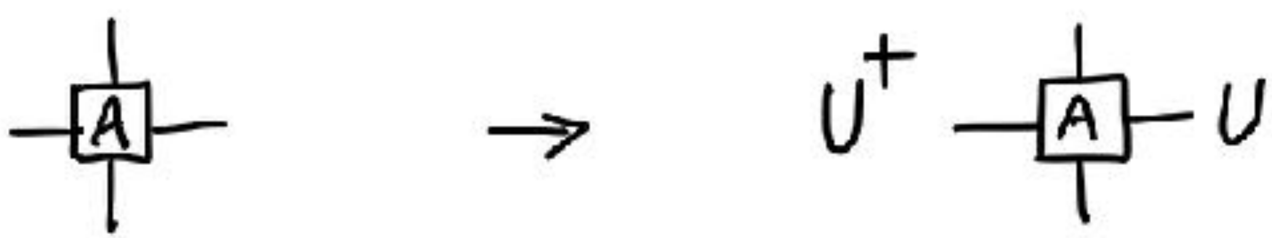
b - scale factor

Some advantages of Tensor RG

- **Local** parametrization of the model - every tensor talks only to nearest neighbors; this locality is preserved under RG

- **Natural norm** - Hilbert-Schmidt

- **More "quantum"**

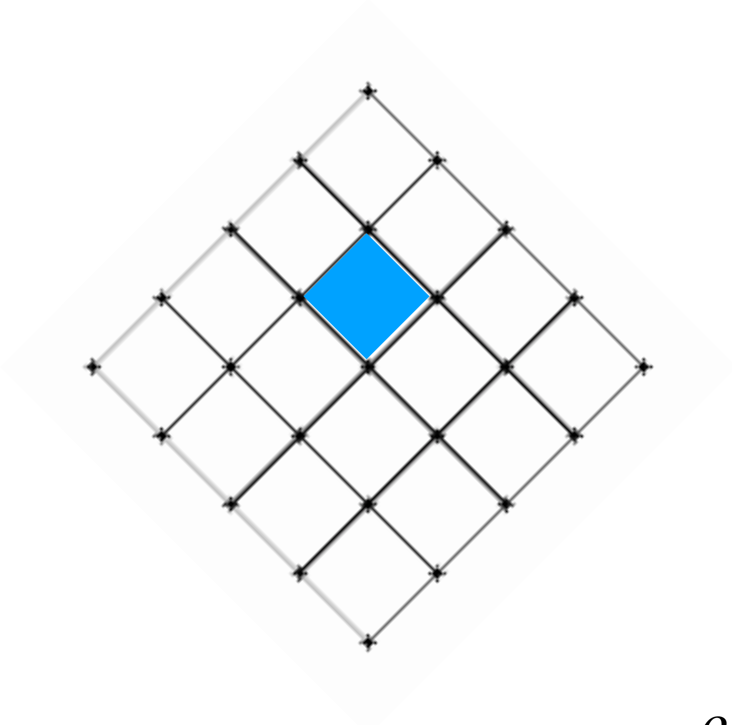


unitary rotation
gives equivalent tensor

This can simplify the tensor.

Also can simplify action of symmetries (transforming to a basis of irreps of symmetry group)

NN Ising example (skip if all clear)



$$H = \beta \sum_{<xy>} s_x s_y, \quad s_x = \pm 1$$

(1) $$A_{ijkl} = e^{\beta(i\cdot j + j\cdot k + k\cdot l + l\cdot i)}$$ $i, j, k, l \in \{\pm 1\}$

(2) Rotate to basis $|0\rangle, |1\rangle = \frac{1}{\sqrt{2}}(|+1\rangle \pm |-1\rangle)$

$A_{0000} = 1$

$$\boxed{\begin{aligned} A_{0011} &= \sinh(4\beta)/(\cosh(4\beta)+3) \\ A_{0101} &= A_{1111} = (\cosh(4\beta)-1)/(\cosh(4\beta)+3) \end{aligned}}$$

go to zero in high-T limit
$\beta \to 0$

Know rigorously:

NN Ising model

$\beta = 0$ $\longrightarrow$ $\beta = \infty$

$A_{0000} = 1$

high-T fixed point A_{HT}

$A_{LT} = A^{(+)} \oplus A^{(-)}$

low-T fixed point
(two copies of HT f.p.)

Thm: (2022, w/ Kennedy)
$\exists$ an analytic tensor-RG map contracts towards A_{HT} in an ε-neighborhood

Involves:
- coarse-graining
- disentangling
- gauge fixing

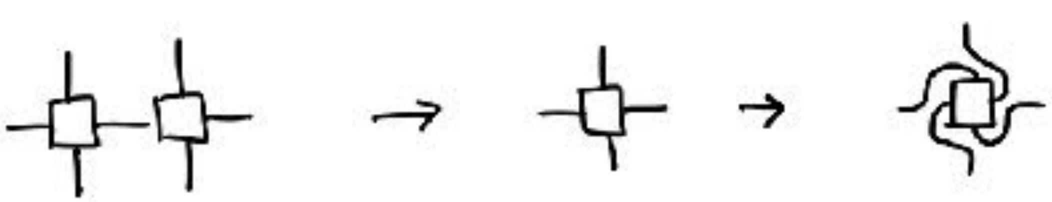
Thm: (2022 w/ Kennedy)
$\exists$ an analytic RG map which contracts towards A_{LT} in an ε-neighborhood preserving Z_2 inv.

- First-order phase transition along the magnetic Z_2-breaking direction.

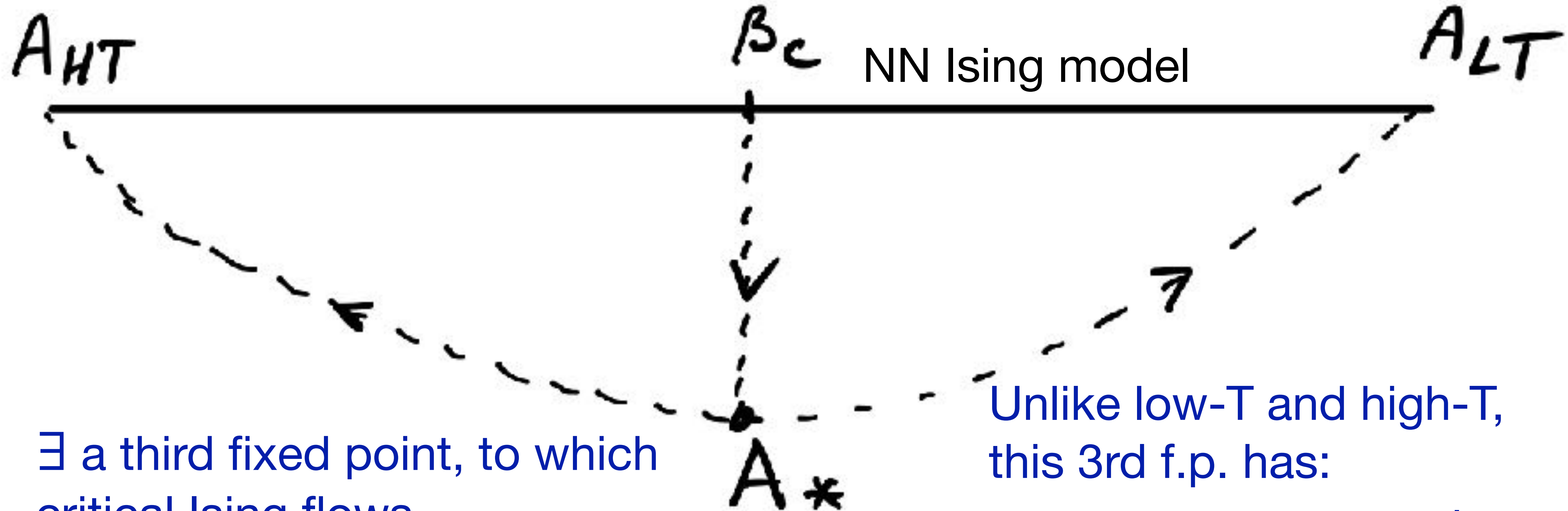
discontinuity fixed point
(Nienhuis, Nauenberg 1975)

2024: Ebel - generalization to 3D (nontrivial, different disentanglers)

2025 (w/Ebel & Kennedy)
- **Quantify convergence neighborhood, computer-assisted**
- **A simpler RG map called 2x1 map**



Expect / Know numerically / Would like to prove:



∃ a third fixed point, to which critical Ising flows

Unlike low-T and high-T, this 3rd f.p. has:

- ∞-many nonzero elements
- one unstable direction

Why do we believe this?

- Many numerical tensor RG algorithms, operate on tensors of finite χ.
- Only approximate - truncation error.
- Truncation error decreases as χ gets larger.
- Conjecture that χ=∞ limit should exist.

Goal & Rules

Prove the existence of the critical fixed point, and get estimates on critical exponents.

We want to do this without using any exact solvability properties of the Ising model in 2D.

In particular we do not use nonlocal transformation to free fermionic representation of NN 2D Ising

This is because the RG method is completely general, and we want to make this clear.

In the future we can imagine that the same ideas will work in 3D.

Difficulties

Numerical algorithms involve some operations on tensors which are a bit difficult to estimate in ∞ - dim context (like eigenvalue and singular value decompositions).

For rigorous analysis, we cannot use the existing algorithms; we have to design our own algorithms.

Newton Method

Since the critical point has one relevant direction, need to set up Newton method.

$$A = R(A)$$ f.p. equation for RG map

$$A = N(A)$$
$$N(A) = A - (I - \nabla R)^{-1}(A - R(A))$$
equivalent f.p. equation for Newton map

Newton map is expected to be a contraction near f.p. provided that ∇R has no eigenvalues 1 in an ε-neighborhood.

At a numerical level, we did this. Ebel,Kennedy,S.R., PRX (2025)

Eigenvalues of ∇R

Two classes of eigenvalues

- **universal** $\lambda_i = b^{D-\Delta_i}$

  where Δ_i are scaling dimension of non-derivative CFT operators

- **non-universal:** λ_i can be anything

  (redundant perturbations ↔ derivative CFT operators)

Our numerical results:

$\mathbb{Z}_2$	$\mathcal{O}$	$\lambda_{\rm CFT} = 2^{2-\Delta_{\mathcal{O}}}$	λ of ∇R
$+$	ϵ	2	1.9996
$+$	T	1	1.0015
$+$	$\overline{T}$	1	0.9980
$+$			0.6322
$+$			$0.5941 \pm 0.1195i$
			$\ldots$
$-$	σ	$2^{15/8} \approx 3.668016$	3.6684
$-$			1.5328
$-$			1.5300
$-$			0.8869
			$\ldots$

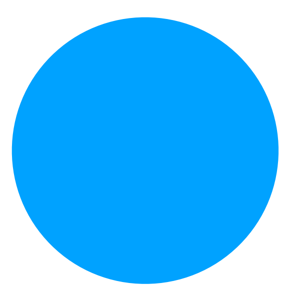
eigs 1

Stress tensor perturbation

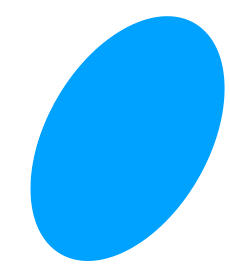
- 2D CFT has **stress-tensor operator** $T_{\mu\nu}$, a symmetric traceless tensor, spin-2 under rotations and scaling dim. 2, i.e. marginal - **eigenvalue 1**

- Perturbing by this operator corresponds to rescaling and rotating the axes

CFT, rotationally invariant



CFT+ $\int h_{\mu\nu} T_{\mu\nu} d^2x$



=> A 2-param family of fixed points!   Newton method ?

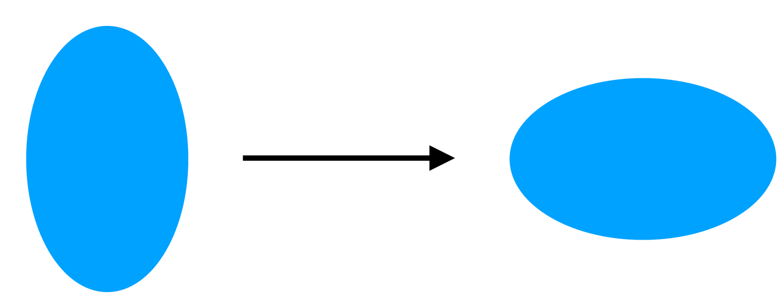
Two strategies:

a) (OPEN) Find tensor RG map which respects spatial symmetries of square lattice (left-right flips and 90° rotations).

b) **OUR IDEA** Modify RG map by adding 90° rotation.

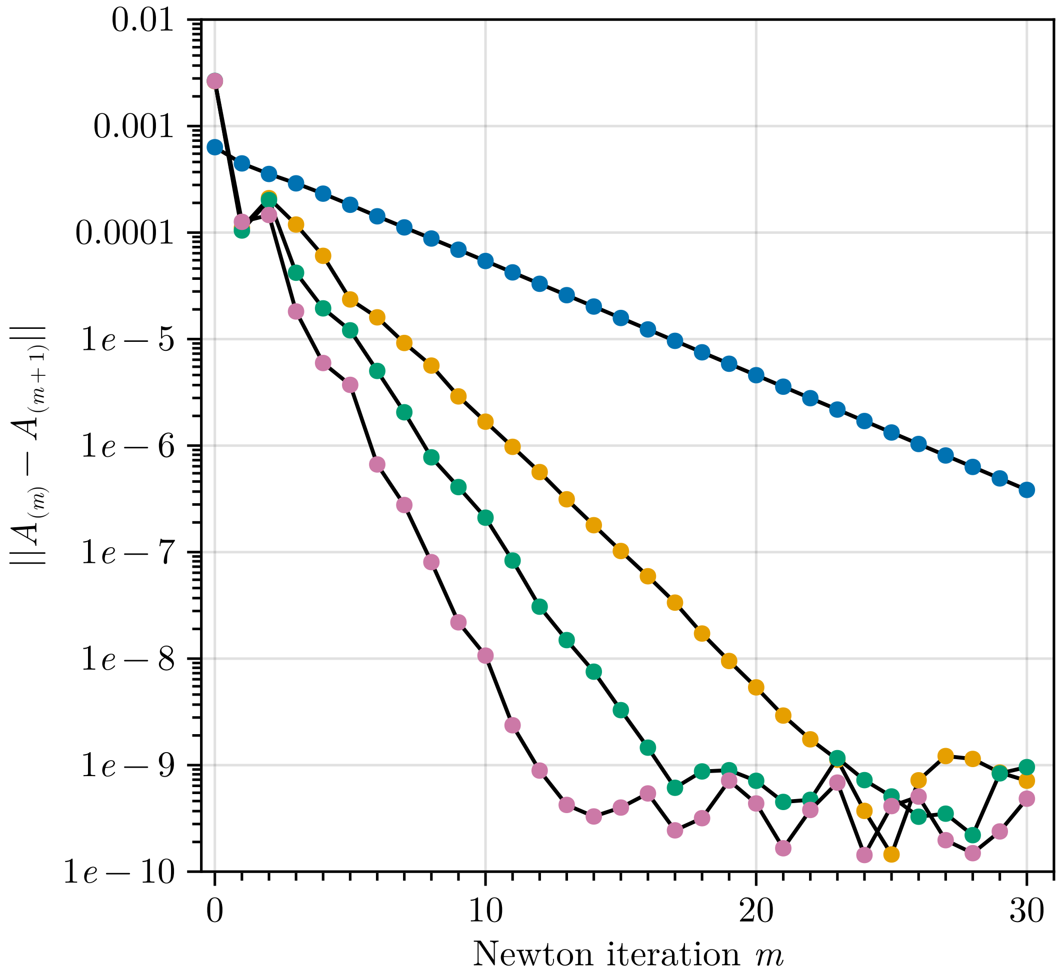
Spin-2 eigenvalues are multiplied by -1:

$1 \to -1$ No problem for Newton



Numerical results for Newton method

Convergence of Newton iterates



Jacobian eigenvalues

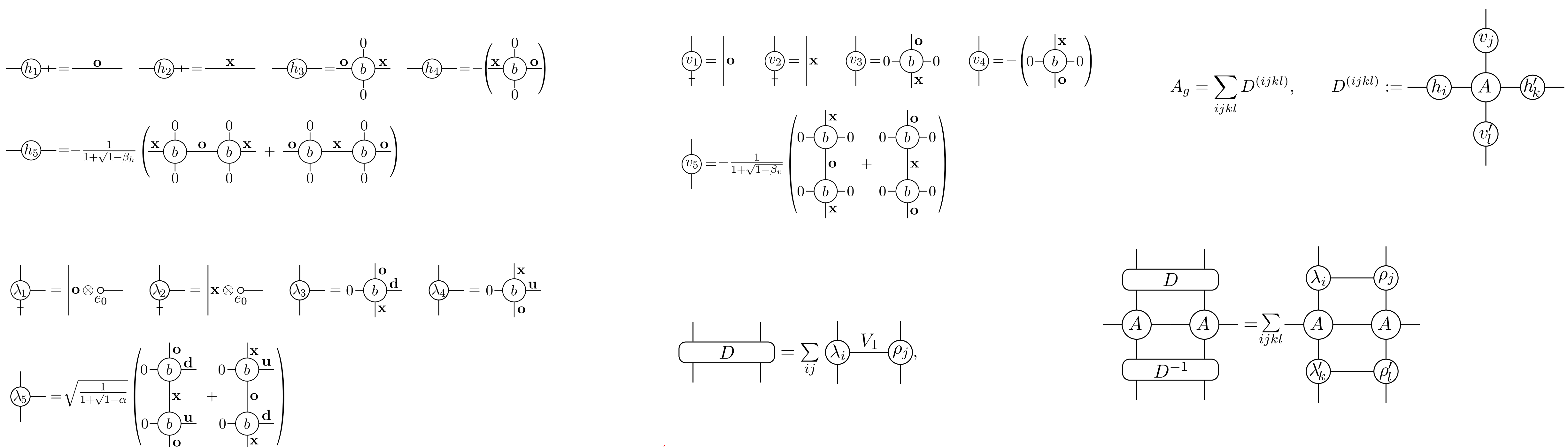
λ of ∇R°
1.9996
-1.0010
-0.9982
0.7757
0.6209
$\ldots$
3.668014
$0.0027 \pm 1.5364i$
0.8600
$0.6318 \pm 0.0583i$
$\ldots$

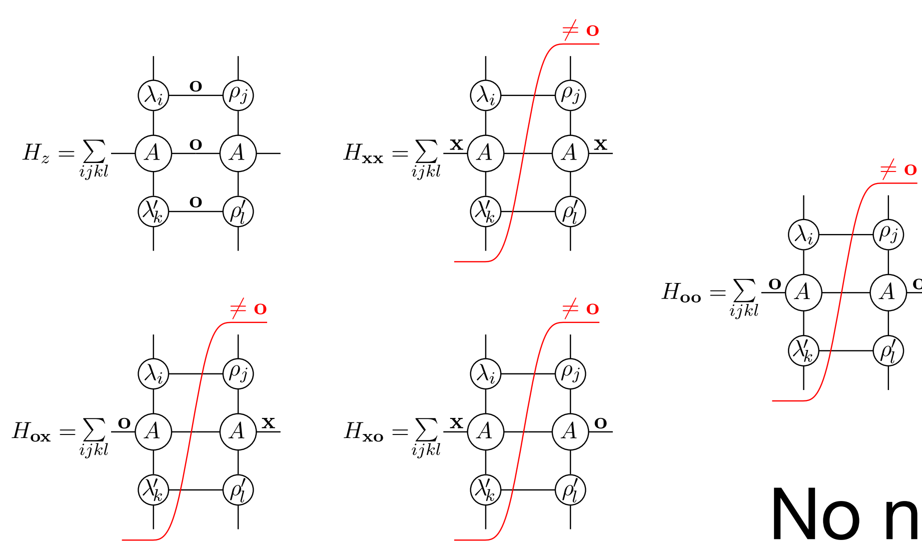
Computer-assisted bounds

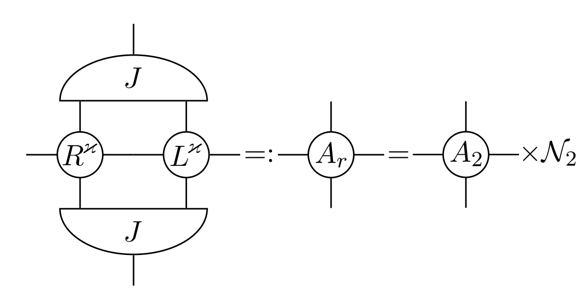
Ebel, Kennedy, S.R., arXiv: 2506.03247

- We define a tensor RG map called a 2x1 map.

- Initial tensor is divided into various pieces, manipulated and recombined using tensor network notation.







No need to understand these equations

Hat-tensor & Master Function

Hat-tensor is a finite-dim "bounding box" for the ∞-dim tensor A.
It measures the norm of various pieces of the tensor:

$$\|A_{i\in a, j\in b, k\in c, l\in d}\| \leq \hat{A}_{abcd}$$

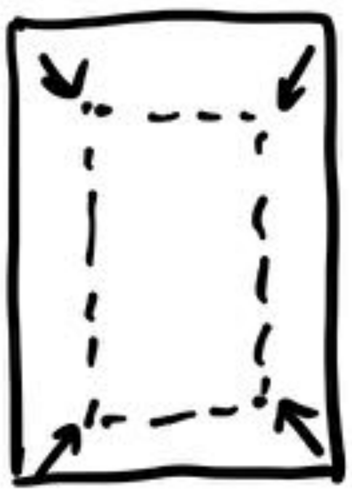
Master function computes how the hat tensor varies from one step to the next.

We get master function by putting hats on all equations defining the RG map (Cauchy-Schwarz)

In our paper we divide the tensor in 63 pieces, so we have a function from $\mathbb{R}^{63}$ to itself, which we evaluate on computer

Earle-Hamilton theorem

Our RG map happens to be analytic (in a complex Hilbert space)



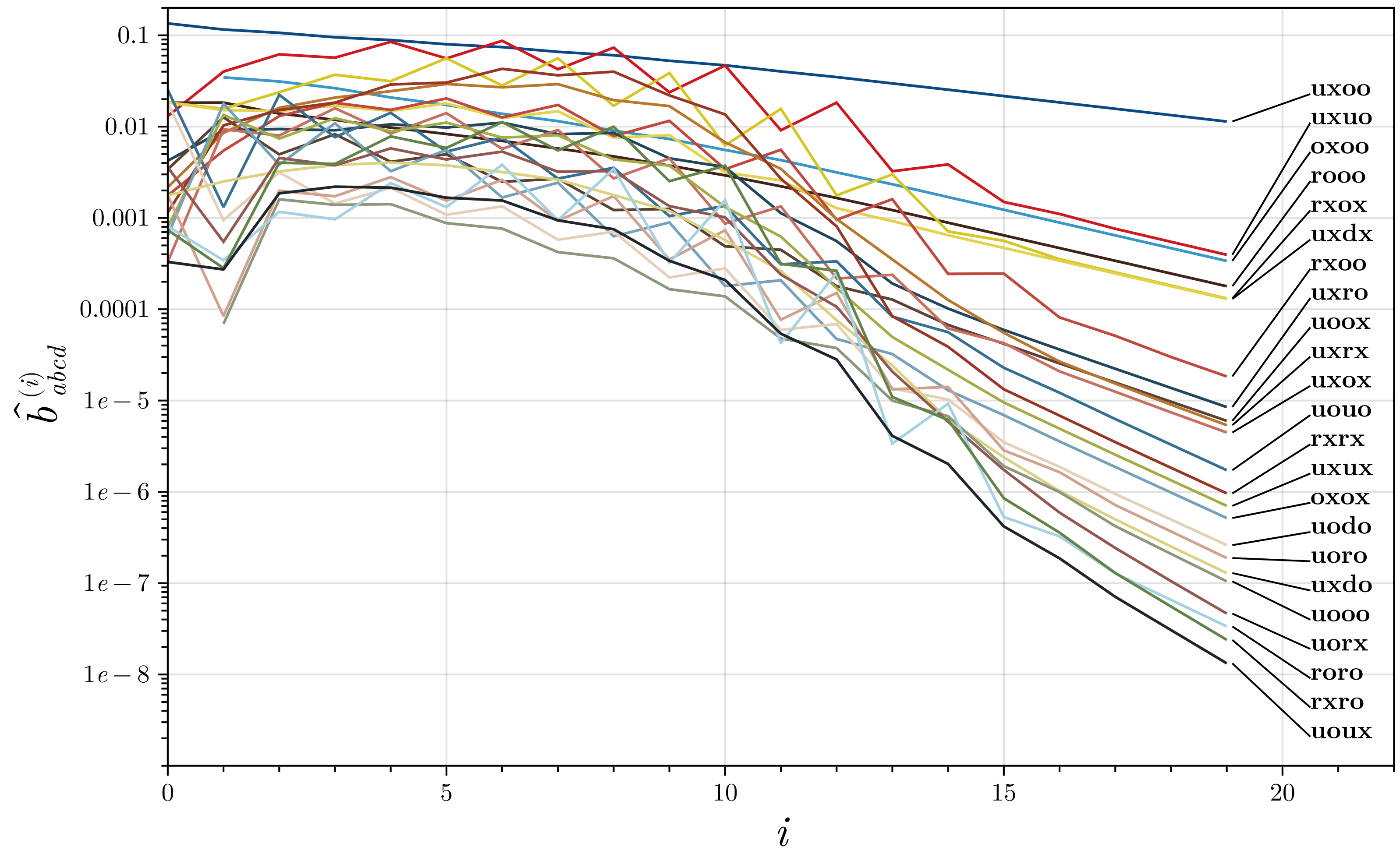
Earle-Hamilton theorem:
if we find a box which is reduced in all directions
then the map is a contraction in the Caratheodory metric

Look for a largest box this condition is verified, with the help of a computer

The same strategy will work around the critical point. Basically, since our map is analytic we only need to bound the map, we don't need to worry about derivatives.

CMP…

Results for XY model



This plot proves that the XY model for $\beta \leq 0.19$ is in the high-T phase

(Numerically $\beta_{BKT} \approx 1.12$)

Conclusions

- Tensor network RG is a new tool for learning about lattice models.

- It allows one to be more quantitative, with computer assistance

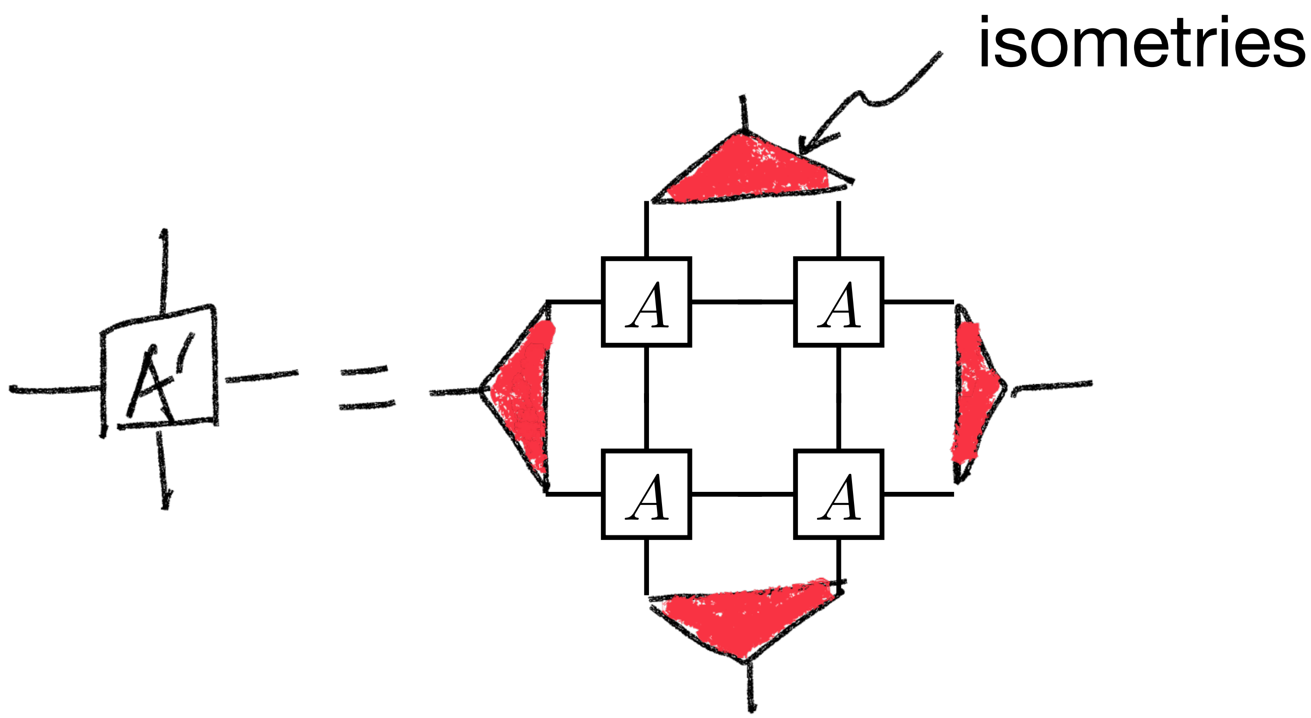
- Nikolay Ebel, Tom Kennedy and myself are busy with constructing the critical fixed point for 2D Ising

- There are countless other possible applications waiting to be explored.

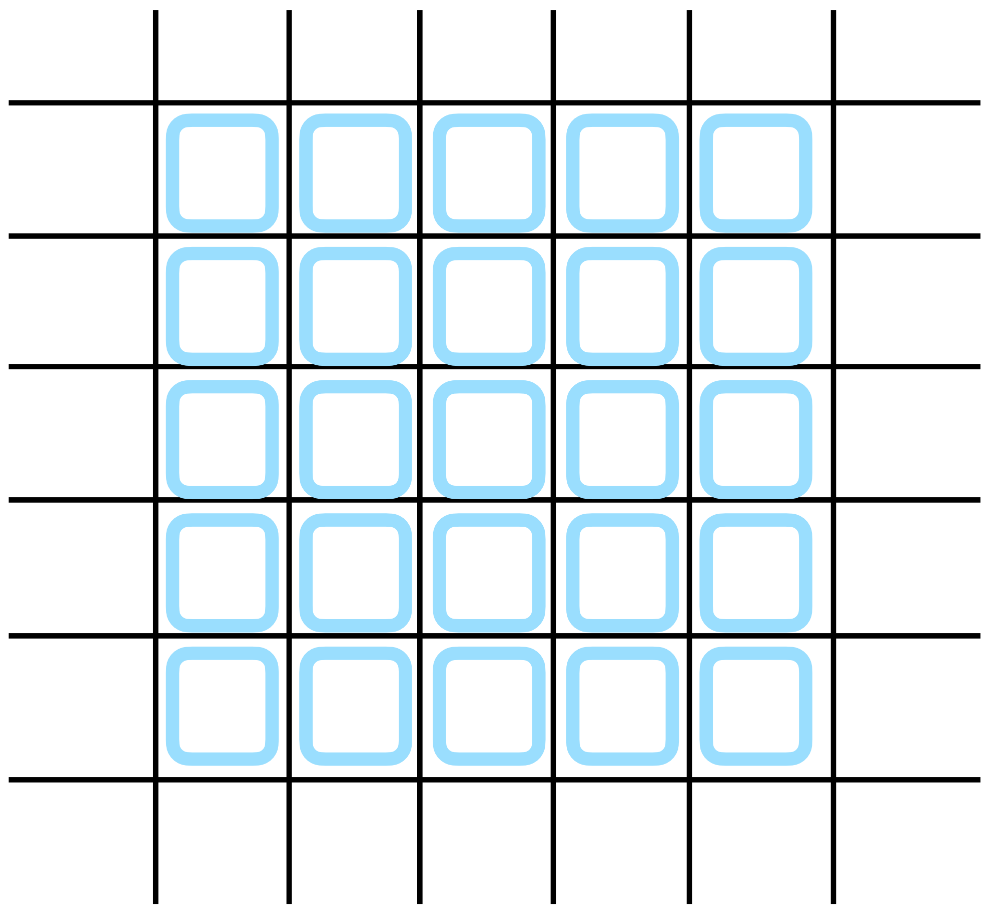
(With Giovanni Rizi (IHES) I am thinking about O(N) NLSM in 2D.)

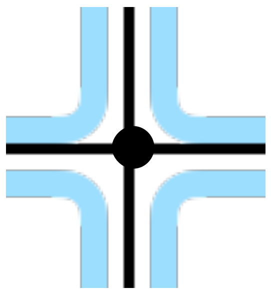
back up

Coarse-graining



With only coarse-graining,
tensor gets polluted by ultra-short correlated degrees of freedom



$A \approx$ 

Disentangling